

A BIVARIATE ANALOGUE TO THE COMPOSED PRODUCT OF POLYNOMIALS

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ABSTRACT. The concept of a *composed product* for univariate polynomials has been explored extensively by Brawley, Brown, Carlitz, Gao, Mills, et al. Starting with these fundamental ideas and utilizing fractional power series representation (in particular, the Puiseux expansion) of bivariate polynomials, we generalize the univariate results. We define a bivariate composed sum, composed multiplication, and composed product (based on function composition). Further, we investigate the algebraic structure of certain classes of bivariate polynomials under these operations. We also generalize a result of Brawley and Carlitz concerning the decomposition of polynomials into irreducibles.

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1. THE UNIVARIATE COMPOSED PRODUCT

To begin, let \mathbb{F}_q denote the finite field of q elements, q a prime power, let $\mathbb{F}_q[x]$ denote the integral domain of polynomials in the indeterminate x over \mathbb{F}_q , and let Γ_q denote the algebraic closure of \mathbb{F}_q . Further, let G denote a nonempty subset of Γ_q which enjoys the following properties:

- (1) G is invariant under the Frobenius automorphism σ where $\sigma : \alpha \mapsto \alpha^q$ for $\alpha \in \Gamma_q$; and
- (2) There is defined on G a binary operation \diamond such that for all $\alpha, \beta \in G$, $\sigma(\alpha \diamond \beta) = \sigma(\alpha) \diamond \sigma(\beta)$.

These properties will be referred to as the \diamond -*properties*.

In the notation of Brawley and Carlitz [3], we denote by $M_G[q, x]$ the set of all nonconstant monic polynomials $f \in \mathbb{F}_q[x]$ whose roots lie in G . Then the *composed product* of the polynomials $f, g \in \mathbb{F}_q[x]$ is defined as

$$(1) \quad (f \odot g)(x) = \prod_{\alpha} \prod_{\beta} (x - \alpha \diamond \beta),$$

where the ordinary products \prod are over all roots α of f and β of g , including multiplicities, so that the degree of $f \odot g$ is the product of the degrees of f and g . (Remark: The notation \diamond has been used in much of the literature to denote both

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the operation on the group G as well as the binary operation on $M_G[q, x]$. In an effort to avoid confusion as well as provide a more natural framework for what is to follow, we have decided to use the \odot operation to denote the composed product operation on $M_G[q, x]$.)

The composed product operation (or \odot -operation) is a binary operation on $M_G[q, x]$ [3], as can be seen by observing that $[(f \odot g)(x)]^q = (f \odot g)(x^q)$. What is also true is that the \odot -operation inherits many of the properties of the binary operation \diamond which induces the \odot -operation. For instance, if G is a semigroup under \diamond then $M_G[q, x]$ is a semigroup under \odot ; if G is commutative under \diamond then $M_G[q, x]$ is commutative under the \odot -operation. If G possesses an identity element $e \in G \cap \mathbb{F}_q$, then the polynomial $x - e$ serves as the identity relative to the \odot -operation. Finally, it is clear that the units of $M_G[q, x]$ are the polynomials $x - u$ where $u \in G \cap \mathbb{F}_q$ is a unit in G .

In their seminal work, Brawley and Carlitz deal with an issue which is analogous to ordinary factorization of polynomials, namely the *decomposition* of a polynomial in $M_G[q, x]$ according to the composed product, where G is now an abelian group under \diamond . We define the following. A polynomial $f \in M_G[q, x]$ which is not a unit is said to be *decomposable* with respect to the \odot -operation if and only if there exist polynomials $g, h \in M_G[q, x]$, each of degree at least two, such that $f(x) = (g \odot h)(x)$. Otherwise, f is said to be *indecomposable*. Polynomials $f, g \in M_G[q, x]$ are said to be *associates* if and only if $f(x) = (r \odot g)(x)$ for some unit $r \in M_G[q, x]$, and in this case we write $f \sim g$. Clearly, \sim is an equivalence relation on $M_G[q, x]$.

Before launching into a detailed discussion of decomposition of polynomials according to \odot , Brawley and Carlitz prove the following theorem which serves to narrow one's focus with regards to the decomposition question.

Theorem 1.1. *Suppose that (G, \diamond) is a group and let $f, g \in M_G[q, x]$ be given with $\deg(f) = m$ and $\deg(g) = n$. Then the composed product $f \odot g$ is irreducible over \mathbb{F}_q if and only if f and g are both irreducible over \mathbb{F}_q with $\gcd(m, n) = 1$.*

Theorem 1.1 encourages us to ask whether irreducibles in $M_G[q, x]$ decompose uniquely, up to associates, into irreducible indecomposables. Brawley and Carlitz show in [3] that the answer to this question is yes when the binary operation on the group G is either ordinary multiplication or ordinary addition, and along the way they provide tests to determine whether a given polynomial is decomposable with respect to either the composed multiplication operation [3], the term used when the operation on G is ordinary multiplication, or the composed sum operation [4], the term used when the operation on G is ordinary addition. Brawley and Brown generalized the work in [3] to include all abelian groups G [2]. The general statement and its corollary are given below.

Theorem 1.2. *Let $G \subset \Gamma_q$ be a group with binary operation \diamond which satisfies the \diamond -properties listed above, let $f(x) \in M_G[q, x]$ be irreducible over \mathbb{F}_q with $\deg(f) = n > 1$, and let \odot denote the composed product operation on $M_G[q, x]$. Suppose that f can be decomposed in two ways as*

$$f(x) = (f_1 \odot f_2 \odot \cdots \odot f_t)(x) = (g_1 \odot g_2 \odot \cdots \odot g_s)(x),$$

where each of the f_i and g_i belong to $M_G[q, x]$ and are each indecomposable with respect to the \odot -operation. Then $s = t$ and there is some reordering of the g_i 's so that $f_i \sim g_i$ for $i = 1, 2, \dots, t$.

We say that $a \in G$ is *indecomposable* with respect to \diamond (G 's binary operation) if the minimal polynomial of a over \mathbb{F}_q cannot be decomposed nontrivially with respect to the composed product operation \odot . This definition provides the following corollary to Theorem 1.2.

Corollary 1.3. *Let $G \subset \Gamma_q$ be a group with binary operation \diamond which satisfies the properties listed above. Every element $\gamma \in G \setminus \mathbb{F}_q$ can be written as a \diamond -product $\gamma = \alpha_1 \diamond \alpha_2 \diamond \dots \diamond \alpha_t$ of a finite number of indecomposables, where $\deg(\gamma)$ over \mathbb{F}_q equals $\prod_{i=1}^t \deg(\alpha_i)$. Moreover, if $\gamma = \beta_1 \diamond \beta_2 \diamond \dots \diamond \beta_s$ is another such \diamond -decomposition of γ , then $s = t$ and there is a reordering of the β_i 's such that for each i from 1 to t , there exists a $c_i \in G \cap \mathbb{F}_q$ such that $\alpha_i = c_i \diamond \beta_i$, and further $c_1 \diamond c_2 \diamond \dots \diamond c_t$ equals the identity element of G .*

Other issues that have been considered in the study of the composed product include the matter of simultaneous decomposition [3], in which it is asked whether a given irreducible can decompose nontrivially (i.e., none of the components in the decomposition is a unit) according to both the composed sum and composed multiplication operations (the answer is no); the efficient computation of a general form of the composed product, and in particular the efficient computation of the composed multiplication and composed sum of two polynomials [5]; the factorization patterns of reducible polynomials $h \in M_G[q, x]$ which decompose as $h(x) = (f \odot g)(x)$ for irreducibles $f, g \in M_G[q, x]$ of non-coprime degrees [10]; and the determination of the group structure of G when the binary operation on G is represented by a bivariate rational function $R(x, y) \in k(x, y) \setminus k[x, y]$, where $k(x, y)$ is the function field in the variables x and y over a field k and $k[x, y]$ is the domain of polynomials in x and y over k [6].

Our goal in this paper is to define a bivariate analogue to the \odot -operation described above. That is, given a field k , indeterminates x and y , and polynomials $f(x, y), g(x, y) \in k[x, y]$ which satisfy certain requirements, we use f and g to create a new polynomial $h(x, y) \in k[x, y]$ in a manner that is similar to the way in which we formed the composed product of two univariate polynomials, as given by (1).

2. FRACTIONAL POWER SERIES.

Our approach to the bivariate case will be to avail ourselves of the well-known theorem of Puiseux (though an earlier version of the theorem was known to Newton).

Theorem 2.1. *Let k be an algebraically closed field and let $f(x, y) = y^m + a_1(x)y^{m-1} + \dots + a_m(x) \in k((x))[y]$ be a monic polynomial with $\deg_y f(x, y) = m > 0$ and coefficients $a_1(x), \dots, a_m(x)$ in $k((x))$. Further, assume either the characteristic of k is zero or that $m!$ is not divisible by the characteristic of k . Then there exists a positive*

integer n , not divisible by the characteristic of k , such that

$$f(x, y) = \prod_{i=1}^m (y - p_i(x^{\frac{1}{n}}))$$

with $p_i(t) \in k((t))$.

Additionally, we have the following corollaries (see [1]).

Corollary 2.2. *If $a_i(x) \in k[[x]]$ for $i = 1, 2, \dots, m$ then $p_i(t) \in k[[t]]$ for $i = 1, 2, \dots, m$.*

Corollary 2.3. *If $f(x, y)$ is irreducible in $k((x))[y]$ then we have $n = m$ (in fact m is the least possible value of n). Moreover, the power series $p_1(x), \dots, p_n(x)$ may be arranged so that $p_i(x^{\frac{1}{n}}) = p_1(\omega^i x^{\frac{1}{n}})$ where ω is a primitive m^{th} -root of unity.*

In letters to Oldenburg, Newton [11] develops a constructive method for determining a $p_i(x)$. This construction forms the basis of the proof of the general theorem, see [12]. It is important to note that Puiseux's theorem is valid in all characteristics. For a more detailed discussion of the characteristic p case see [8].

3. THE BIVARIATE COMPOSED PRODUCT

3.1. Definitions. We are now prepared to generalize the univariate case. For $f(x, y), g(x, y) \in k((x))[y]$, we invoke Theorem 2.1 to write $f(x, y) = \prod_{i=1}^{m_1} (y - p_i(x^{\frac{1}{n_1}}))$ and $g(x, y) = \prod_{j=1}^{m_2} (y - q_j(x^{\frac{1}{n_2}}))$. We now make the following definitions.

Definition 3.1. *The composed sum of $f(x, y)$ and $g(x, y)$ is given by*

$$f \star g = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} (y - (p_i(x^{\frac{1}{n_1}}) + q_j(x^{\frac{1}{n_2}})))$$

Definition 3.2. *The composed multiplication of $f(x, y)$ and $g(x, y)$ is given by*

$$f \bullet g = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} (y - (p_i(x^{\frac{1}{n_1}}) q_j(x^{\frac{1}{n_2}})))$$

In the bivariate case we have an additional operation available to us, namely function composition.

Definition 3.3. *If $f(0, 0) = g(0, 0) = 0$ (f and g have 0 as constant term) then let the composed product of $f(x, y)$ and $g(x, y)$ be given by*

$$f \odot g = \prod_{i=1}^{m_1} \prod_{j=1}^{m_2} (y - p_i(q_j(x^{\frac{1}{n_2}})^{\frac{1}{n_1}}))$$

We quickly illustrate each of these operations with an example (for computational convenience, we assume we are working over \mathbb{C}). Let $f(x, y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$. Then, applying Newton's construction method we obtain $y = p(x^{\frac{1}{4}}) = x^{\frac{6}{4}} + x^{\frac{7}{4}}$ as one solution (see [7]). We observe that f is irreducible so by Corollary 2.3, we have $p_i(x^{\frac{1}{4}}) = (\omega_1^i x^{\frac{1}{4}})^6 + (\omega_1^i x^{\frac{1}{4}})^7 = p(\omega_1^i x^{\frac{1}{4}})$ for $i = 1, \dots, 4$, where ω_1 is a primitive fourth root of unity. Hence,

$$f(x, y) = \prod_{i=1}^4 (y - p(\omega_1^i x^{\frac{1}{4}})).$$

Similarly, let $g(x, y) = y^6 - 3x^3y^4 - 2x^5y^3 + 3x^6y^2 - 6x^8y - x^9 + x^{10}$. Like f , g is irreducible. In this case, Newton's construction method yields a root $y = q(x^{\frac{1}{6}}) = x^{\frac{9}{6}} + x^{\frac{10}{6}}$. Hence, $q_j(x^{\frac{1}{6}}) = (\omega_2^j x^{\frac{1}{6}})^9 + (\omega_2^j x^{\frac{1}{6}})^{10} = q(\omega_2^j x^{\frac{1}{6}})$ for $j = 1, \dots, 6$. Here, ω_2 is a primitive sixth root of unity. Thus,

$$g(x, y) = \prod_{j=1}^6 (y - q(\omega_2^j x^{\frac{1}{6}})).$$

Using these factorizations we compute the composed sum to be

$$\begin{aligned} f \star g &= \prod_{i=1}^4 \prod_{j=1}^6 (y - (p(\omega_1^i x^{\frac{1}{4}}) + q(\omega_2^j x^{\frac{1}{6}}))) \\ &= \prod_{i=1}^4 \prod_{j=1}^6 (y - (\omega_1^{6i} x^{\frac{6}{4}} + \omega_1^{7i} x^{\frac{7}{4}} + \omega_2^{9j} x^{\frac{9}{6}} + \omega_2^{10j} x^{\frac{10}{6}})). \end{aligned}$$

A similar computation shows the composed multiplication is

$$\begin{aligned} f \bullet g &= \prod_{i=1}^4 \prod_{j=1}^6 (y - (p(\omega_1^i x^{\frac{1}{4}})q(\omega_2^j x^{\frac{1}{6}}))) \\ &= \prod_{i=1}^4 \prod_{j=1}^6 (y - (\omega_1^{6i} \omega_2^{9j} x^3 + \omega_1^{7i} \omega_2^{9j} x^{\frac{13}{4}} + \omega_1^{6i} \omega_2^{10j} x^{\frac{19}{6}} + \omega_1^{7i} \omega_2^{10j} x^{\frac{41}{12}})) \end{aligned}$$

The composed product is, on the other hand, a much more complicated computation. Then, using our example f and g we get (after expanding the terms as power series and simplifying)

$$\begin{aligned} f \odot g &= \prod_{i=1}^4 \prod_{j=1}^6 (y - p(\omega_1^i q(\omega_2^j x^{\frac{1}{6}})^{\frac{1}{4}})) \\ &= \prod_{i=1}^4 \prod_{j=1}^6 (y - (\omega^{12i+6j} x^{\frac{54}{24}} + \frac{3}{2} \omega^{12i+10j} x^{\frac{58}{24}} + \frac{3}{8} \omega^{12i+14j} x^{\frac{62}{24}} + \omega^{18i+15j} x^{\frac{63}{24}} + \dots)) \end{aligned}$$

where ω represents a primitive 24^{th} root of unity.

3.2. Properties. In what follows, \mathbb{M} will represent the set of all polynomials in $k((x))[y]$ which satisfy the hypotheses of Theorem 2.1, while $\mathbb{M}^* \subset \mathbb{M}$ will represent the set of all elements of \mathbb{M} whose Puiseux expansions have roots p_i which belong to $k((x)) \setminus k$ for all i .

It is natural to ask what sort of structure \mathbb{M} possesses under the \star , \bullet , and \odot operations. When the characteristic of k is zero, we have the following.

Theorem 3.4. (a) (\mathbb{M}, \star) is a commutative semigroup with identity $e(x, y) = y$. (b) (\mathbb{M}, \bullet) is a commutative semigroup with identity $e(x, y) = y - 1$. (c) (\mathbb{M}^*, \odot) is a non-commutative semigroup with identity $e(x, y) = y - x$.

Proof. We prove (a), as the proofs of (b) and (c) are similar. Select any pair of elements $f(x, y), g(x, y) \in \mathbb{M}$, with $\deg_y f(x, y) = m$ and $\deg_y g(x, y) = n$; then clearly $(f \star g)(x, y) \in \mathbb{M}$ as $f \star g$ is monic in y with $\deg_y(f \star g) = mn > 0$ and with coefficients in $k((x))$, and thus closure under \star is proved. The associativity and commutativity of the \star operation follow from the associativity and commutativity properties which $k((x))$ enjoys under ordinary addition. The identity element of (\mathbb{M}, \star) is clearly $e(x, y) = y$, while the identity elements for (\mathbb{M}, \bullet) and (\mathbb{M}, \odot) are $e(x, y) = y - 1$ and $e(x, y) = y - x$ respectively. \square

Note that in general we cannot make the same statements when the characteristic is $p > 0$, for we can always find $f(x, y), g(x, y) \in \mathbb{M}$ whose degrees in y are m and n respectively such that $p \mid (mn)!$. However, if one considers the set of all elements of \mathbb{M} whose degree in y is 1 (we denote this by \mathbb{M}_1), as well as the set of elements of \mathbb{M}_1 of the form $f(x, y) = y - a(x)$ with $a(x) \in k((x)) \setminus k$ (denoted by \mathbb{M}_1^*), we have the following.

Theorem 3.5. Let \mathbb{M}_1 be defined as above with $\text{char}(k) \geq 0$. Then $(\mathbb{M}_1, \star, \bullet)$ is a field with \star -identity $e(x, y) = y$ and \bullet -identity $i(x, y) = y - 1$. Further, $(\mathbb{M}_1, \star, \bullet) \cong (k((x)), +, \cdot)$.

Proof. The proof that $(\mathbb{M}_1, \star, \bullet)$ is a field proceeds in the same fashion as the proof of Theorem 3.4.

To see that $(\mathbb{M}_1, \star, \bullet) \cong (k((x)), +, \cdot)$ we consider the canonical map $\varphi : \mathbb{M}_1 \rightarrow k((x))$ via $\varphi : (y - a(x)) \mapsto a(x)$. Note that if $f(x, y) = y - a(x)$ and $g(x, y) = y - b(x)$ then $\varphi(f(x, y) \star g(x, y)) = \varphi(y - (a(x) + b(x))) = a(x) + b(x) = \varphi(f(x, y)) + \varphi(g(x, y))$. Similarly, $\varphi(f(x, y) \bullet g(x, y)) = \varphi(f(x, y)) \cdot \varphi(g(x, y))$. Thus φ is a homomorphism of fields. If $\varphi(y - a(x)) = \varphi(y - b(x))$ then, clearly, $a(x) = b(x)$; hence, φ is injective. For any $a(x) \in k((x))$ we have $a(x) = \varphi(y - a(x))$, so φ is also surjective. \square

For the \odot -operation, we have the following.

Theorem 3.6. (\mathbb{M}_1^*, \odot) is a non-commutative semigroup with identity $e(x, y) = y - x$.

Proof. Proceeds in the same fashion as the proof of Theorem 3.4. \square

For certain subsets of \mathbb{M}^* under \odot with $\text{char}(k) = p > 0$, a somewhat greater structure can be obtained. Specifically, let $\mathbb{M}_{h,n}$ denote the set of all elements of \mathbb{M}^* which are homogeneous polynomials of degree $0 < n < \sqrt{p}$ and whose x^n

and y^n coefficients are nonzero (the reason for the second requirement will become obvious shortly), and set $\mathbb{M}_h = \bigcup_{n=1}^{\lfloor \sqrt{p} \rfloor} \mathbb{M}_{h,n}$, where $\lfloor s \rfloor$ is the greatest integer less than or equal to s . Using Newton's expansion method for determining the Puiseux series, it is easy to show that for any $f(x, y) = \sum_{i+j=n} a_{i,j} x^i y^j \in \mathbb{M}_h$, a Puiseux expansion of f is given by $y = \alpha x$ where α is a root of the polynomial $w_f(t) = a_{0,n} t^n + a_{1,n-1} t^{n-1} + \cdots + a_{n-1,1} t + a_{n,0}$, $a_{n,0} \neq 0$ (so the zero element of k is not a root of w_f ; this will be important when we employ the \odot -operation on a certain subset of \mathbb{M}_h). We call w_f the *polynomial associated with f* .

Let $\mathbb{M}_{h,min}$ denote the set of elements $f(x, y) \in \mathbb{M}_h$ whose associated polynomials w_f act as their own minimum polynomials. That is, if $a \in k$, $r(t) \in k[t]$ with $w_f(a) = 0 = r(a)$ then $w_f(t) \mid r(t)$ in $k[t]$. Thus, $w_f(t)$ is irreducible over some subfield of k ; specifically, $w_f(t)$ is irreducible over L where $L \subset k$ is the smallest subfield containing the coefficients of $w_f(t)$. Now let $k = \mathbb{F}_q$ and let $f, g \in \mathbb{M}_{h,min}$ with $\deg(f) = m$, $\deg(g) = n$, $(m, n) = 1$, and the coefficients of f and g belonging to \mathbb{F}_q . Further let the associated polynomials of f and g be w_f and w_g , respectively, with roots a and b respectively. Then $F(x, y) = (f \odot g)(x, y)$ is a homogeneous polynomial of degree $mn < p$ with associated polynomial w_F having root ab . Since $(m, n) = 1$ with w_f and w_g irreducible we see by Theorem 1.1 that $w_F = w_f \odot w_g$ (where (G, \odot) is a multiplicative group here) is an irreducible univariate polynomial of degree mn over \mathbb{F}_q with root ab . Thus, $(f \odot g)(x, y) \in \mathbb{M}_{h,min}$ and we have the following result.

Lemma 3.7. *Let k be an algebraically closed field having characteristic p , and let $f(x, y), g(x, y) \in \mathbb{M}_{h,min}$ be homogeneous polynomials whose associated polynomials $w_f(t), w_g(t) \in k[t]$ have coprime degrees. Then the \odot -operation, when performed on f and g , is a well-defined operation, that is $F(x, y) = (f \odot g)(x, y)$ is a homogeneous polynomial of degree $\deg(f)\deg(g)$ whose associated polynomial w_F has degree $\deg(w_f)\deg(w_g)$. Moreover, $(f \odot g) \odot h = f \odot (g \odot h)$ for all triplets $f(x, y), g(x, y), h(x, y) \in \mathbb{M}_{h,min}$ whose associated polynomials have coprime degrees, and also $f \odot g = g \odot f$ for all pairs $f(x, y), g(x, y) \in \mathbb{M}_{h,min}$ whose associated polynomials have coprime degrees. The polynomial $e(x, y) = y - x$ serves as the identity element relative to the \odot -operation.*

From Lemma 3.7 we have the following.

Theorem 3.8. *Let k be an algebraically closed field having characteristic p , and let $\mathbb{M}_{h,min,1} \subset \mathbb{M}_{h,min}$ denote the set of elements of $\mathbb{M}_{h,min}$ of degree 1. Then $(\mathbb{M}_{h,min,1}, \odot)$ is an abelian group. Further, $(\mathbb{M}_{h,min,1}, \odot) \cong (k^*, \cdot)$.*

Proof. Select any pair of elements $f(x, y), g(x, y) \in \mathbb{M}_{h,min,1}$, and write their Puiseux expansions as $y = ax$ and $y = bx$ respectively (where $a, b \neq 0$ and are each of degree 1 over k). Under the \odot -operation, the Puiseux expansion for the composed product of f and g is $y = (ab)x$. Thus $(f \odot g)(x, y)$ has associated polynomial $w(t)$ which is of degree one, and hence is the minimum polynomial of ab over k . The \odot -operation is clearly associative and commutative on $\mathbb{M}_{h,min,1}$, and it is clear as well that the identity element of $\mathbb{M}_{h,min,1}$ under \odot is given by $e(x, y) = y - x$. As for inverses, we

see that if $f(x, y), g(x, y) \in \mathbb{M}_{h, \min}$ have Puiseux expansions $y = ax$ and $y = a^{-1}x$ respectively then $(f \odot g)(x, y) = e(x, y)$.

If $f(x, y) \in \mathbb{M}_{h, \min, 1}$ then $f(x, y) = y - ax$, $a \in k^*$. Let $\varphi : \mathbb{M}_{h, \min, 1} \rightarrow k^*$ by $\varphi : (y - ax) \mapsto a$. Let $f(x, y) = y - ax$ and $g(x, y) = y - bx$. We have $\varphi(f(x) \odot g(x)) = \varphi(y - abx) = ab = \varphi(f(x)) \cdot \varphi(g(x))$ showing φ is a homomorphism. If $\varphi(f(x, y)) = \varphi(y - ax) = \varphi(y - bx) = \varphi(g(x, y))$ then $a = b$ so $f(x, y) = g(x, y)$ making φ injective. For every $a \in k^*$ we have $a = \varphi(y - ax)$, hence φ is surjective. Thus φ is an isomorphism. \square

What is particularly nice about the elements of $\mathbb{M}_{h, \min}$ is that their definition, along with the definition of the \odot -operation applied to these polynomials, gives rise to a bivariate analogue of Theorem 1.2 in the case where the coefficients of the elements of $\mathbb{M}_{h, \min}$ come from $\mathbb{F}_q((x)) \subset \Gamma_q((x))$. To see this, we first adapt the definition of associates given in Section 2. Specifically, two polynomials $f(x, y), g(x, y) \in \mathbb{M}_{h, \min}$ are *associates* of each other under the \odot -operation provided there exists a polynomial $h(x, y) = y - ax \in \mathbb{F}_q[x, y]$ such that $f(x, y) = (h \odot g)(x, y)$. To denote this relationship, we write $f \sim g$, and note that this relation is an equivalence relation on $\mathbb{M}_{h, \min}$. We have the following.

Theorem 3.9. *Let $F(x, y)$ be a homogeneous polynomial of degree $n < p$ in $\mathbb{F}_q((x))[y]$ whose associated polynomial acts as the minimum polynomial for its roots, and let \odot denote the composed product operation on $\mathbb{M}_{h, \min}$. Suppose that F decomposes over $\mathbb{M}_{h, \min}$ as*

$$F(x, y) = (f_1 \odot f_2 \odot \cdots \odot f_t)(x, y)$$

where $f_i(x, y) \in \mathbb{M}_{h, \min}$ for each $i = 1, \dots, t$ and $(\deg(f_i), \deg(f_j)) = 1$ for $i \neq j$. Suppose that $F(x, y)$ can be decomposed in an alternate fashion as follows:

$$F(x, y) = (g_1 \odot g_2 \odot \cdots \odot g_s)(x, y),$$

where $g_i(x, y) \in \mathbb{M}_{h, \min}$ for each i from 1 to s and $(\deg(g_i), \deg(g_j)) = 1$ for $i \neq j$. Then $s = t$ and there is some reordering of the g_i 's so that $f_i \sim g_i$ for i from 1 to t .

Proof. The Puiseux expansion of F is $y = ax$ for some element $a \in \Gamma_q$ of degree n over \mathbb{F}_q . Since $f(x, y)$ decomposes as $F(x, y) = (f_1 \odot f_2 \odot \cdots \odot f_t)(x, y)$, it follows from the definition of the \odot -operation that a can be written as $a = a_1 a_2 \cdots a_t$ where $y = a_i x$ is a Puiseux expansion of f_i for each i . Likewise since $F(x, y)$ decomposes as $F(x, y) = (g_1 \odot g_2 \odot \cdots \odot g_s)(x, y)$, it follows that a can be written as $a = b_1 b_2 \cdots b_s$ where $y = b_i x$ is a Puiseux expansion of g_i for each i . By Corollary 1.3, $s = t$ and there is a reordering of the b_i 's so that $a_i = c_i b_i$ for each i from 1 to t where $c_i \in \mathbb{F}_q$ for each i and $c_1 c_2 \cdots c_t = 1$. Thus for each i there exists a polynomial $h_i(x, y) \in \mathbb{M}_{h, \min, 1}$ such that $f_i(x, y) = (h_i \odot g_i)(x, y)$, that is $f_i \sim g_i$ for i from 1 to t . \square

4. SUMMARY

We have reviewed some fundamental notions regarding both the composed product of univariate polynomials over a finite field and Puiseux expansions for plane curves, and have combined key aspects of each subject to form a bivariate counterpart to the composed product operation. The definition of the bivariate composed

product is not dependent upon the characteristic of the field from which the coefficients come. Examples of bivariate composed products of various forms have been given, and some properties of this new composition have been introduced, particularly for the case in which the algebraic closure has positive characteristic. In particular, we have shown that the decomposition theorem for the univariate composed product has an analogue in the bivariate case for certain homogeneous polynomials in $\mathbb{F}_q[x, y]$.

The question of whether there is a suitable multivariate (here, three or more variables) counterpart to the composed product is still open. Gonzalez, Getino, and Farto have recently produced a multivariate extension of the Newton-Puiseux algorithm [9], but the algorithm relies on a monomial ordering compatible with the order of the terms of a Taylor series expansion, and thus one cannot guarantee that the resulting Puiseux expansion will be unique. Note: The authors have partially addressed this matter in a subsequent paper, see [11]

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